

# Comparative aspects of spin-dependent interaction potentials for spin-1/2 and spin-1 matter fields

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## Abstract

This paper sets out to establish a comparative study between classes of spin- and velocity-dependent potentials for spin-1/2 and spin-1 matter currents/sources in the non-relativistic regime. Both (neutral massive) scalar and vector particles are considered to mediate the interactions between (pseudo-)scalar sources or (pseudo-)vector currents. Though our discussion is more general, we contemplate specific cases in which our results may describe the electromagnetic interaction with a massive (Proca-type) photon exchanged between two spin-1/2 or two spin-1 carriers. We highlight the similarities and peculiarities of the potentials for the two different types of charged matter and also focus our attention on the comparison between the particular aspects of two different field representations for spin-1 matter particles. We believe that our results may contribute to a further discussion of the relation between charge, spin and extensibility of elementary particles.

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## I. INTRODUCTION

Back to 1950, the paper by Matthews and Salam [1] and the subsequent works by Salam in 1952 [2] clarified the quantum field-theoretical approach to the electrodynamics of (massive) charged scalar particles. Ever since, the problem of extending this investigation to include the case of (massive) charged vector particles became mandatory in view of the theoretical evidence for the role of charged and neutral spin-1 particles that should couple to the charged and neutral currents.

From 1960, Komar and Salam [3], Salam [4, 5], Lee and Yang [6] and Delbourgo and Salam [7], gave a highly remarkable push for the understanding and construction of a fundamental theory for the microscopic interaction between charged vector bosons and photons. Further works by Tzou [8], Aronson [9] and Velo and Zwanziger [10] summed up efforts to the previous papers and the final conclusion was that a consistent unitary and renormalizable quantum field-theoretical model would be possible only in a non-Abelian scenario with a Higgs sector that spontaneously breaks the gauge symmetry to give mass to the vector bosons, without spoiling the unitarity bounds for the cross section of the scattering of longitudinally polarized charged vector bosons [11].

Here, we adopt this framework: the electrodynamics of (massive) charged vector bosons with a non-minimal dipole-type term coupling to the gauge field. In the case of the massless photon, this ensures a  $g = 2$ -value for the g-ratio of spin-1 particles. We should however point out that this dipole-type interaction is non-minimal from the viewpoint of an Abelian symmetry; if we take into account the local  $SU(2)$  symmetry that backs the plus- and minus-charged vector bosons, the dipole coupling in the action is actually a minimal (non-Abelian,  $SU(2)$ ) interaction term after spontaneous symmetry breaking has taken place.

In the present paper, our main effort consists in pursuing a detailed investigation of the semi-classical aspects of the charge and spin interactions for massive charged matter of a vector nature. Our central purpose is to compare the features and specific profiles of the influence of the spin of the charge carriers on interaction potentials (electromagnetic or a more general  $U(1)$  interaction) between two different categories of sources/currents: fermionic and spin-1 bosonic. We shall present the conclusions of our comparative procedure at the end of our calculations.

At this point, we highlight that the literature in the topic contemplates a great deal of articles discussing the structure of the electromagnetic current and electrodynamical aspects of spin-1 charged matter [12]-[22]. We pursue an investigation of an issue not considered in connection with spin-1 charged matter: the spin- and velocity-dependence of the interaction potential associated with (pseudo-)scalar sources and pseudo-vector currents that interact by exchanging scalar and vector mediators, respectively. For the spin-1, these specific cases have not been addressed to in the literature. These extra sources/currents may not necessarily be associated to the electromagnetic interaction in that they do not follow from the  $U(1)$  symmetry of the electromagnetism. We may be describing a new force between these extra sources/current whose origin could be traced back to some more fundamental physics. The case of the usual vector current is reassessed here and our results match with the ones in the literature. It is worthy mentioning that fermionic sources/currents can display a wide range of spin- and velocity-dependent interparticle potentials [23][24][25], and several of them have been reconsidered in this article.

The electrodynamics of ordinary fermionic matter is very well understood, from the macroscopic to the quantum level. Scalar and vector bosonic charged matter, on the other hand, experience a richer variety of interactions when coupled to the electromagnetic field. So far, most of the theoretical and experimental literature dealing with macroscopic interactions consider only spin-1/2 matter, i.e., fermionic sources/currents [26]-[29]. This preference is naturally due to the fact that ordinary charges in matter are carried by electrons. Elementary spin-1 charged particles are difficult to observe, since the only known examples are the  $W^\pm$  gauge bosons, whose mass is too large - and lifetime too short - to allow direct inspection (its full width is  $\Gamma_W \sim 2 \text{ GeV}$  [30]). Though not stable enough to be directly handled, elementary spin-1 particles have, in principle, their own electrodynamics and it is of theoretical interest to study how it deviates, or not, from its fermionic counterpart. On the other hand, at the atomic and nuclear level, it would be a good motivation to have a spin- and velocity-dependent expression for the electromagnetic potential between ionised spin-1 (charge =  $\pm 1$ ) atoms and charged spin-1 nuclei or hypernuclei [31][32].

We highlight here a particular feature of bosonic charge carriers as far as the electromagnetic interactions are concerned. From a purely macroscopic point of view, the Maxwell equations address the problem of determining field configurations from given charge and current densities and a number of duly specified boundary conditions. They do not take

into account the microscopic nature of these sources ( $\rho$  and  $\vec{j}$ , respectively). If a microscopic description of charged matter in terms of classical fields is given (based on a local  $U(1)$  symmetry), the particular aspects of the charge carriers become salient and London-type terms [33] may arise. We shall be more specific about this point at the final Section of our paper, where we render more evident the peculiarities of the spin of the charge carriers in the electromagnetic and general Abelian interactions.

Our paper is outlined as follows: in Section II, we introduce the concept of potential and briefly discuss the kinematics involved, as well as present the notations and conventions employed. In Section III, we calculate the sources/currents and potentials, discussing the similarities between spin-1/2 and spin-1 matter. We consider, in Section IV, another possible field representation for the spin-1 charged carrier, namely via a rank-2 tensor field, and we also work out the corresponding source/current–source/current interaction potentials. We present our conclusions and perspectives in Section V. Natural units are adopted throughout, where  $\hbar = c = 1$ , and we assume  $\epsilon^{0123} = +1$ .

## II. METHODOLOGY

Since we are interested in comparing the low-energy behavior of spin-1/2 and spin-1 as the interacting matter sector, it is convenient to work in the non-relativistic (NR) limit of the interactions. For simplicity we choose to work in the center of mass (CM) reference frame, in which particle 1 has initial and final 3-momenta given by  $\vec{P} = \vec{p} - \vec{q}/2$  and  $\vec{P}' = \vec{p} + \vec{q}/2$ , respectively. Here,  $\vec{p}$  is the average momentum of particle 1 before and after interaction, while  $\vec{q}$  is the momentum transfer carried by the intermediate boson - see Fig.(1).

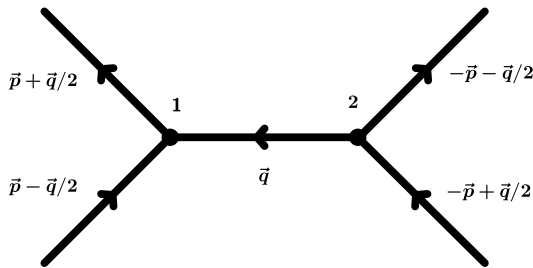


FIG. 1. Momenta assignments for the tree-level interaction between sources/currents 1 and 2.

The potential is calculated through the first Born approximation [34], i.e. the Fourier transform of the amplitude,  $\mathcal{A}(\vec{q}, m\vec{v})$ , with respect to the momentum transfer

$$V(r) = - \int \frac{d^3\vec{q}}{(2\pi)^3} \mathcal{A}(\vec{q}, \vec{p}) e^{i\vec{q}\cdot\vec{r}}, \quad (1)$$

where we explicitly assume that the amplitude contains terms dependent on the velocity of particle 1. It is clear from the NR approximations of the sources/currents presented below that not only velocity-dependent terms will arise in the potentials, but also spin-dependent ones. This is due to our choice of keeping terms in the currents which go beyond the zeroth order, thus static, case [35].

In what follows, we shall consider only elastic interactions, which, together with energy conservation, imply  $q^0 = 0$  and  $\vec{p} \cdot \vec{q} = 0$ , pointing out that  $q$  is a space-like 4-vector:  $q^2 = -\vec{q}^2$ . Furthermore, in the amplitude we will keep only terms up to second order in  $|\vec{p}|/m_{1,2}$ .

The Feynman rules for the tree-level diagram above are equivalent to taking  $iJ_{1,2}$  as the interaction vertices, where  $J_{1,2}$  are the matter sources/currents associated with particles 1 and 2, respectively. The corresponding amplitude may then be written as

$$\mathcal{A} = iJ_1 \langle \text{prop.} \rangle J_2, \quad (2)$$

where  $\langle \text{prop.} \rangle$  is the momentum-space propagator of the intermediate boson with adequate Lorentz indices as the case may be. In this paper, we are interested in interactions mediated by massive neutral scalars (Klein-Gordon-type) and spin-1 vector particles (Proca-type), whose momentum-space propagators are:

$$\langle \phi\phi \rangle = \frac{i}{q^2 - m_\phi^2} \quad (3)$$

and

$$\langle A_\mu A_\nu \rangle = \frac{-i}{q^2 - m_A^2} \left( \eta_{\mu\nu} - \frac{q_\mu q_\nu}{m_A^2} \right). \quad (4)$$

The sources/currents  $J_1$  and  $J_2$ , as representatives of the respective vertices, must translate the different possible couplings to the gauge sector: scalar, pseudo-scalar, vector and pseudo-vector. The vector currents are obtained from the Lagrangian through the Noether method, but, for the bosonic case, we consider only first order in the electromagnetic coupling constant,  $e$ , since we wish to compute  $\mathcal{A}$  to second order in  $e$ . We shall discuss this in more details in Subsection B. The other sources/currents are built based on the principle

that they should be bilinear in the field and its complex conjugate and reflect the desired symmetry.

Below, we recall some basic properties from spin-1/2 and spin-1 starting from the rest frame where the essential degrees of freedom become apparent, and then Lorentz-boosting it to the LAB frame. This will be important when we calculate the NR limit of each source/current and then apply them to extract the interparticle potential. Here, we choose to work the spin-1 in its vector field representation, but, in Section IV, we shall discuss it in its less usual tensor field representation [19][20][36].

### A. Spin-1/2

For the sake of completeness, we present the well-known properties of Dirac fermions. The Dirac equation for the positive-energy spinors is, in momentum-space,

$$[\gamma^\mu P_\mu - m] u(P) = 0 \quad (5)$$

with the corresponding equation for the spinor conjugate,  $\bar{u} \equiv u^\dagger \gamma^0$ , with the attribution  $\bar{u} = \bar{u}(P')$ . By using the gamma matrices in the Dirac representation, the positive-energy spinor can be explicitly written in the NR limit as

$$u(P) \simeq N_f \begin{pmatrix} \xi \\ \frac{\vec{\sigma} \cdot \vec{P}}{2m} \xi \end{pmatrix}, \quad (6)$$

with  $\xi$  are the basic spinors satisfying  $\xi_r^\dagger \xi_s = \delta_{rs}$  and  $N_f$  denotes a normalization constant. Manipulating eq.(5) and its conjugate, we obtain the Gordon decomposition of the vector current,

$$\bar{u} \gamma^\mu u = \frac{p^\mu}{m} \bar{u} u + \frac{i}{m} q_\nu \bar{u} \Sigma^{\mu\nu} u \quad (7)$$

expressed in terms of the more convenient variables  $p \equiv (P + P')/2$  and  $q \equiv P' - P$ , where we defined the spin matrix as  $\Sigma^{\mu\nu} \equiv \frac{i}{4} [\gamma^\mu, \gamma^\nu]$ . Similar decompositions are also possible for other types of bilinear forms: for example, using eq.(5) and its conjugate, we show that

$$q_\mu \bar{u} \gamma^\mu \gamma_5 u = 2m \bar{u} \gamma_5 u. \quad (8)$$

The Gordon decomposition of the vector current associated with the Noether theorem, eq.(7), yields a first term proportional to the density of the field ( $\sim |u|^2$ ), while the second

one presents a coupling between momentum transfer,  $q$ , and the spin matrix,  $\Sigma^{\mu\nu}$ , of the respective representation. This connection will be responsible for the zeroth order magnetic dipole moment interaction of the lepton with gyromagnetic ratio  $g = 2$  [37]. Interestingly enough, this simple structure found for the Gordon decomposition for the spin-1/2 will have a perfect analog in the spin-1 case, if we correctly include a non-minimal coupling - see Subsection B.

Our interparticle potentials shall be derived from the (effective) Lagrangian describing the interactions between the exchanged (scalar or vector) particles and the sources/currents of the fermionic particle. It reads as follows below:

$$\mathcal{L}_{int}^{s=1/2} = \bar{\psi} [(g_S + g_{PS}i\gamma_5) \phi + (g_V\gamma^\mu + g_{PV}\gamma^\mu\gamma_5) A_\mu] \psi, \quad (9)$$

where the different couplings are all dimensionless. The scalar (S) and pseudo-scalar (PS) sources are not conserved and, since we are dealing with a massive Dirac field, we also have a non-conserved pseudo-vector (PV) current (see eq.(8)). As pointed out by Boulware [38], the Proca-type field, whenever coupled to a conserved current, leads to a renormalizable model. For this reason, we shall consider the coupling between the  $PV$ - current and Proca-type field as an effective interaction. The UV divergence does not harm our purposes here, since we are interested in the low-energy regime described by the interparticle potential. Actually, our external (on-shell) sources/currents are tailored in the NR regime, meaning that we are working much below the scale where UV divergences show up. Henceforth we shall denote the coupling constant  $g_V \equiv e$  whenever considering the electromagnetic interaction.

## B. Spin-1

In the previous Subsection, we have revised some basic aspects concerning spin-1/2 particles. Here, we would like to follow up with a similar discussion for massive charged spin-1 particles in order to comparatively study how differently (or not) spin-1/2 and spin-1 sources/currents interact in the NR limit.

We start off discussing the vector field representation for a massive charged spin-1 particle, whose dynamics can be obtained from the following Lagrangian:

$$\mathcal{L}_{vec.} = -\frac{1}{2} \mathcal{W}_{\mu\nu}^* \mathcal{W}^{\mu\nu} + m^2 W_\mu^* W^\mu + ie(g-1) W_\mu^* W_\nu F^{\mu\nu} \quad (10)$$

where  $m$  is the mass and  $\mathcal{W}_{\mu\nu}$  the gauge-covariant field strength, given by  $D_\mu W_\nu - D_\nu W_\mu$ , with  $D_\mu = \partial_\mu + ieA_\mu$ . The last term in eq.(10) is a non-minimal coupling between the bosonic matter fields and the field-strength of the interaction mediator and its importance lies in the fact that it asserts that  $g = 2$  in tree-level for the spin-1 particle [39][40][41] (see also references therein).

The field  $W^\mu = (W^0, \vec{W})$  has, in principle, four degrees of freedom and these are reduced through the constraints introduced by the equations of motion, which read

$$D_\mu \mathcal{W}^{\mu\nu} + m^2 W^\nu - ie(g-1)W_\mu F^{\mu\nu} = 0 \quad (11)$$

with analogue ones for the complex conjugated field. Our goal is to characterize the source/current– source/current interaction and for this, at tree-level, the aforementioned free source/current are to be described by bilinears in the spin-1 matter fields, where the derivatives are the ordinary ones. In other words, since the asymptotic states are the particle states composed only of  $W$  fields (we are now taking the free field equations), we have the subsidiary condition  $\partial_\mu W^\mu = 0$ . This constraint is better seen in momentum-space, where we have  $\partial_\mu = -iP_\mu$ , so that

$$W^0 = \frac{1}{E} \vec{P} \cdot \vec{W}, \quad (12)$$

where, in order to avoid a cumbersome notation, we have used the same symbol to denote the field in position- and momentum-space, as there is no risk of confusion. This equation relates different components of the vector field and shows that the time-component is proportional to the longitudinal projection of  $\vec{W}$  and that  $W^0$  is first-order in velocity, playing the role of a “small” component similar to the spin-1/2 case. It also allows us to write  $W^\mu$  in its rest frame as  $W_{rest}^\mu = N_W (0, \vec{\epsilon})$ , where  $\vec{\epsilon}$  stands for the dimensionless polarization 3-vector in the rest frame of the free particle and  $N_W$  is a normalization constant.

The next step is to bring the system from rest into motion (i.e., to the LAB frame) via an appropriate Lorentz transformation. In doing so, we obtain

$$W_{lab}^\mu = N_W \left[ \frac{1}{m} (\vec{P} \cdot \vec{\epsilon}), \vec{\epsilon} + \frac{1}{m(E+m)} (\vec{P} \cdot \vec{\epsilon}) \vec{P} \right] \quad (13)$$

and with this, two comments are in order:

- Eq.(12) tells us that, in the rest frame (where  $\vec{P} = 0$ ), all the information about the vector field is contained in its spatial part. In this frame the only 3-vector available is the polarization, i.e., its spin, which again enforces the vector character of the field.



- The normalization constant is given by  $N_W = 1/\sqrt{2m}$ .

We wish to make some considerations on the vector currents, both global and local, associated to the Lagrangian (10). At the global stage, i.e., prior to the gauging of the  $U(1)$  symmetry, the Noether current (in configuration-space) reads

$$J_{global}^\mu = ie (W^{*\mu\nu} W_\nu - W^{\mu\nu} W_\nu^*), \quad (14)$$

where no covariant derivative is involved, so  $W_{\mu\nu} \equiv \partial_\mu W_\nu - \partial_\nu W_\mu$ . Upon the gauging of the symmetry, and by including the non-minimal coupling  $g$ -term, the current changes into

$$J_V^\mu = ie (\mathcal{W}^{*\mu\nu} W_\nu - \mathcal{W}^{\mu\nu} W_\nu^*) + ie(g-1)\partial_\nu (W^\nu W^{*\mu} - W^{*\nu} W^\mu). \quad (15)$$

Now, an important remark concerning current conservation: to calculate the current-current potential at the tree-level approximation (one-boson exchange), as we are doing here, we actually only consider the current to the first order in  $e$ . This amounts to neglecting the terms in the gauge potential present in the current in eq.(15). In other words: to calculate the potential to order  $e^2$ , we consider the current only up to order  $e$ . Therefore, to the desired order in the coupling constant, the vector current is the globally conserved one, plus the non-minimal  $g$ -term. At this perturbative level, the current  $J_V^\mu$  is given by the global and non-minimal terms, as displayed in the second line of eq.(16) below:

$$\begin{aligned} \mathcal{L}_{int}^{s=1} = & \left[ g_S W_\mu^* W^\mu + g_{PS} W_{\mu\nu}^* \tilde{W}^{\mu\nu} \right] \phi + \\ & + ig_V [W^{*\mu\nu} W_\nu - W^{\mu\nu} W_\nu^* + (g-1)\partial_\nu (W^\nu W^{*\mu} - W^{*\nu} W^\mu)] A_\mu + \\ & + ig_{PV} [\tilde{W}^{*\mu\nu} W_\nu - \tilde{W}^{\mu\nu} W_\nu^*] A_\mu, \end{aligned} \quad (16)$$

where the coupling constants have the following canonical mass dimensions:  $[g_S] = -[g_{PS}] = 1$  and  $[g_V] = [g_{PV}] = 0$ , and the interactions between the sources/currents and the mediating (scalar or vector) bosons are all displayed.

By using the equations of motion and the (free) subsidiary conditions, we may re-write the vector current in momentum space as

$$J_V^\mu(p, q) = 2ep^\mu W_\nu^* W^\nu + iegq_\nu W^{*\alpha} (\Sigma_V^{\mu\nu})_{\alpha\beta} W^\beta \quad (17)$$

in which  $(\Sigma_V^{\mu\nu})_{\alpha\beta} \equiv i (\delta_\alpha^\mu \delta_\beta^\nu - \delta_\alpha^\nu \delta_\beta^\mu)$  stands for the spin matrix in the  $(\frac{1}{2}, \frac{1}{2})$ -representation of the Lorentz group and, as for the spin=1/2 case, we have set  $g_V \equiv e$ . In the equation above

we left the  $g$ -factor explicit but in the NR limits we shall use its tree-level value,  $g = 2$ . Eq.(17) is nothing but the Gordon decomposition for the vector field representation of a massive and charged spin-1 and the similarity with the spin-1/2 case, eq.(7), is remarkable. This Gordon decomposition for spin-1 particles was found independently and agrees with the results of Ref.[19].

We denote the dual of  $W_{\mu\nu}$ , as given in the  $PS$ - and  $PV$ - couplings in eq.(16), by means of  $\tilde{W}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} W^{\alpha\beta}$ . We also emphasize that the (global)  $PV$ - bosonic current is conserved in a topological sense, i.e., without making use of either the equations of motion or the symmetries of the Lagrangian. As in the case of spin-1/2, the  $S$ - and  $PS$ - sources for spin-1 are equally not conserved.

In the next Section, we discuss the momentum-space sources/currents for spin-1/2 and spin-1 in the NR limit and calculate the interparticle potentials generated by the exchange of scalar and vector mediating bosons.

### III. NON-RELATIVISTIC CURRENTS AND POTENTIALS

Following the indications above, we present below the NR limit of the sources/currents, which may be extracted (in configuration-space) from Lagrangians (9) and (16) as  $J\phi$  or  $J_\mu A^\mu$ , as the case may be. In the following, the fields are already normalized with  $N_f = 1$  and  $N_W = 1/\sqrt{2m}$ . From now on, for the sake of clarity, we shall adopt the variables  $\{p, q\}$  instead of  $\{P, P'\}$ , so  $u(P) = u(p - q/2)$  and  $\bar{u}(P') = \bar{u}(p + q/2)$ ; similar definitions hold for  $W_\mu(P)$  and  $W_\mu^*(P')$ . We point out that the matter sources/currents developed in this Section are for particle 1, i.e., the left vertex in Fig.(1). The second source/current may be obtained by taking  $\vec{q} \rightarrow -\vec{q}$ ,  $\vec{p} \rightarrow -\vec{p}$  in the first one.

At this point it is convenient to reinforce the difference between polarization and spin of the particle. In the fermionic case, we shall denote the expectation value of its spin,  $\sigma_i$ , by contracting the basic spinors  $\xi$  with the Pauli matrices, namely,  $\langle \sigma_i \rangle \equiv \xi^\dagger \sigma_i \xi$ . The same idea is applied to the bosonic case, where the spin matrix of the particle is  $(S_i)_{jk} = i\epsilon_{ijk}$ , with  $\Sigma_{ij} = \epsilon_{ijk} S_k$ , and its expectation value is given by  $\langle S_i \rangle \equiv \vec{\epsilon}_j^* (S_i)_{jk} \vec{\epsilon}_k$ .

A remark about our notation: in the process of calculating the amplitude factors, the form  $\xi^\dagger \xi \equiv \delta_i$  and  $\vec{\epsilon}^* \cdot \vec{\epsilon} \equiv \delta_i$ , with  $i = 1, 2$  labeling the particle, will appear frequently.

These indicate a possible spin-flip between initial and final states, and we shall leave the  $\delta$ 's explicit in the final expressions, despite the fact that, in general, low-energy interactions will not induce a change in the spin orientation of the particles involved.

### A. Scalar (Klein-Gordon type) exchange

By means of the relations presented in the previous sections, we can carry out the NR expansion of the  $S$ - and  $PS$ - of the spin-1 sources. For the  $S$ -source of spin-1/2 particles, we can use the Dirac spinor, eq.(6), and its conjugate, to obtain

$$J_{S(s=1/2)} = g_S^1 \left\{ \delta_1 \left[ 1 - \frac{1}{4m_1^2} \left( \vec{p}^2 - \frac{1}{4}\vec{q}^2 \right) \right] - \frac{i}{4m_1^2} \vec{q} \cdot (\vec{p} \times \langle \vec{\sigma} \rangle_1) \right\}. \quad (18)$$

For the  $S$ -source of the spin-1 particles, we obtain the expression below:

$$W_\mu^* W^\mu = \frac{g_S^1}{2m_1} \left\{ -\delta_1 + \frac{1}{2m_1^2} \left[ (\vec{p} \cdot \vec{\epsilon}_1) (\vec{q} \cdot \vec{\epsilon}_1^*) - (\vec{p} \cdot \vec{\epsilon}_1^*) (\vec{q} \cdot \vec{\epsilon}_1) - (\vec{q} \cdot \vec{\epsilon}_1) (\vec{q} \cdot \vec{\epsilon}_1^*) \right] \right\}, \quad (19)$$

which can be written in a more enlightening way if we note that the polarization and momenta 3-vectors satisfy  $-i(\vec{q} \times \vec{p}) \cdot \langle \vec{S} \rangle = (\vec{p} \cdot \vec{\epsilon}) (\vec{q} \cdot \vec{\epsilon}^*) - (\vec{p} \cdot \vec{\epsilon}^*) (\vec{q} \cdot \vec{\epsilon})$ . This allows us to set the spin-1 scalar source as

$$J_{S(s=1)} = -\frac{g_S^1}{2m_1} \left\{ \delta_1 + \frac{1}{2m_1^2} \left[ i(\vec{q} \times \vec{p}) \cdot \langle \vec{S} \rangle_1 + (\vec{q} \cdot \vec{\epsilon}_1) (\vec{q} \cdot \vec{\epsilon}_1^*) \right] \right\}. \quad (20)$$

By proceeding analogously, we can find the pseudo-scalar sources listed below

$$J_{PS(s=1/2)} = -\frac{ig_{PS}^1}{2m_1} \vec{q} \cdot \langle \vec{\sigma} \rangle_1 \quad (21)$$

$$J_{PS(s=1)} = ig_{PS}^1 \vec{q} \cdot \langle \vec{S} \rangle_1, \quad (22)$$

and here we note that both pseudo-scalar sources have the same functional form characterized by the coupling between spin and momentum transfer. In the following, we shall present the potentials.

Let us calculate a particular case, namely that of  $S - S$  interaction, for both  $s = 1$  and  $s = 1/2$  sources. We start off with the fermionic case. As discussed in the Methodology, the general amplitude is given by eq.(2) and, in this case,

$$\mathcal{A}_{S-S} = \frac{i}{q^2 - m_\phi^2} J_1 J_2, \quad (23)$$

where we have used the Klein-Gordon propagator, eq.(3), and the momentum assignments of Fig.(1):

$$J_1 \equiv g_S^1 \bar{u}(p + q/2) u(p - q/2), \quad (24)$$

$$J_2 \equiv g_S^2 \bar{u}(-p - q/2) u(-p + q/2). \quad (25)$$

At this point, we use the fermionic scalar source presented in eq.(18). This scalar source remains unchanged by taking  $q \rightarrow -q$  and  $p \rightarrow -p$ , since it is quadratic in the momenta (it will only be necessary a change in labels:  $1 \rightarrow 2$ ). If we keep terms up to second order in the momenta, the amplitude becomes

$$\begin{aligned} \mathcal{A}_{S-S}^{s=1/2} \simeq & \frac{g_S^1 g_S^2}{\vec{q}^2 + m_\phi^2} \left\{ \delta_1 \delta_2 \left[ 1 - \frac{1}{4} \left( \vec{p}^2 - \frac{\vec{q}^2}{4} \right) \left( \frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \right] + \right. \\ & \left. - \frac{i}{4} (\vec{q} \times \vec{p}) \cdot \left[ \frac{\delta_1 \langle \vec{\sigma} \rangle_2}{m_2^2} + \frac{\delta_2 \langle \vec{\sigma} \rangle_1}{m_1^2} \right] \right\} \end{aligned} \quad (26)$$

and the Fourier transformation given in eq.(1) yields the potential:

$$\begin{aligned} V_{S-S}^{s=1/2} = & -\frac{g_S^1 g_S^2}{4\pi} \frac{e^{-m_\phi r}}{r} \left\{ \delta_1 \delta_2 \left[ 1 - \frac{1}{4} \left( \vec{p}^2 + \frac{m_\phi^2}{4} \right) \left( \frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \right] + \right. \\ & \left. + \frac{(1 + m_\phi r)}{4r^2} \vec{L} \cdot \left( \frac{\delta_1 \langle \vec{\sigma} \rangle_2}{m_2^2} + \frac{\delta_2 \langle \vec{\sigma} \rangle_1}{m_1^2} \right) \right\}. \end{aligned} \quad (27)$$

As expected, the dominant term is the monopole-monopole given by the Yukawa interaction,  $-\delta_1 \delta_2 e^{-m_\phi r} / (4\pi r)$ . We also obtain second-order corrections to this term in the form of a monopole-dipole with spin-orbit couplings.

The amplitude of the bosonic  $S - S$  case presents the same structure as given in eq.(23), but with the sources below:

$$J_1 \equiv g_S^1 W_\mu^*(p + q/2) W^\mu(p - q/2), \quad (28)$$

$$J_2 \equiv g_S^2 W_\mu^*(-p - q/2) W^\mu(-p + q/2). \quad (29)$$

We follow the same program, i.e., by using the NR limit of the bosonic scalar source, eq.(20), we obtain

$$\mathcal{A}_{S-S}^{s=1} \simeq \frac{g_S^1 g_S^2}{4m_1 m_2 (\vec{q}^2 + m_\phi^2)} \left\{ \delta_1 \delta_2 + \left[ \frac{\delta_1}{2m_2^2} \left[ i (\vec{q} \times \vec{p}) \cdot \langle \vec{S} \rangle_2 + (\vec{q} \cdot \vec{\epsilon}_2^*) (\vec{q} \cdot \vec{\epsilon}_2) \right] + 1 \leftrightarrow 2 \right] \right\}$$

and, performing the Fourier integral, we get

$$V_{S-S}^{s=1} = -\frac{g_S^1 g_S^2}{16\pi m_1 m_2} \frac{e^{-m_\phi r}}{r} \left\{ \delta_1 \delta_2 - \frac{(1 + m_\phi r)}{2r^2} \vec{L} \cdot \left( \frac{\delta_1 \langle \vec{S} \rangle_2}{m_2^2} + \frac{\delta_2 \langle \vec{S} \rangle_1}{m_1^2} \right) + \right. \\ \left. + \zeta \left[ \frac{\delta_1}{2m_2^2 r^2} \left[ \delta_2 (1 + m_\phi r) - (\hat{r} \cdot \vec{\epsilon}_2) (\hat{r} \cdot \vec{\epsilon}_2^*) (3 + 3m_\phi r + m_\phi^2 r^2) \right] + 1 \leftrightarrow 2 \right] \right\}, \quad (30)$$

where the parameter  $\zeta$  in the last term may assume the values  $\zeta = \pm 1$ ; in eq.(30) its value is  $\zeta = 1$ . However, in the particular situation described in Section IV (a 2-form description of the massive spin-1 particle), the potential (30) also appears, but with  $\zeta = -1$ .

Here we notice that this potential describes the same interactions as in the fermionic case, namely, the Yukawa factor and a spin-orbit term, but it also displays a “polarization-polarization” interaction term. This observation will be present in other situations in our comparison between the fermionic and bosonic potentials. Below, we quote the results for the other bosonic potentials:

$$V_{S-PS}^{s=1} = \frac{g_S^1 g_{PS}^2 \delta_1}{8\pi m_1} \frac{(1 + m_\phi r) e^{-m_\phi r}}{r^2} \hat{r} \cdot \langle \vec{S} \rangle_2 \quad (31)$$

$$V_{PS-PS}^{s=1} = \frac{g_{PS}^1 g_{PS}^2}{4\pi} \frac{e^{-m_\phi r}}{r} \left\{ \left( m_\phi^2 + \frac{3m_\phi}{r} + \frac{3}{r^2} \right) (\hat{r} \cdot \langle \vec{S} \rangle_1) (\hat{r} \cdot \langle \vec{S} \rangle_2) + \right. \\ \left. - \left( \frac{m_\phi}{r} + \frac{1}{r^2} \right) \langle \vec{S} \rangle_1 \cdot \langle \vec{S} \rangle_2 \right\}. \quad (32)$$

The fermionic  $PS - PS$  and  $S - PS$  potentials have the same functional form as the bosonic ones and, if we neglect a contact term (that is, a  $\delta^3(\vec{r})$  term), these results are very similar to those obtained by Moody and Wilczek [24] in the context of fermionic sources exchanging axions. Indeed, they only differ by a mass factor due the canonical mass dimension of our coupling constants.

## B. Massive vector (Proca type) exchange

Firstly, let us focus on the bosonic case, more precisely, on the spin-1 vector current,  $J_V^\mu$ . For the  $\mu = 0$  component in eq.(17), we have, in momentum-space

$$J_{V(s=1)}^0 = -e_1 \left[ \delta_1 + \delta_1 \left( \frac{\vec{p}^2}{2m_1^2} + \frac{\vec{q}^2}{8m_1^2} \right) \right] + \\ + (g - 1) \frac{e_1}{2m_1^2} \left[ (\vec{p} \cdot \vec{\epsilon}_1^*) (\vec{q} \cdot \vec{\epsilon}_1) - (\vec{q} \cdot \vec{\epsilon}_1^*) (\vec{p} \cdot \vec{\epsilon}_1) + (\vec{q} \cdot \vec{\epsilon}_1^*) (\vec{q} \cdot \vec{\epsilon}_1) \right], \quad (33)$$

where the entire second line comes from the  $g$ -factor correction term. If  $g = 1$ , that line vanishes and we would have an incomplete description of the electromagnetic interaction experienced by the vector boson, as seen from eq.(10). Rearranging the equation above and taking the correct  $g = 2$ -value, we obtain the time-component of the spin-1 vector current in the form

$$J_{V(s=1)}^0 = -e_1 \delta_1 \left( 1 + \frac{\vec{p}^2}{2m_1^2} + \frac{\vec{q}^2}{8m_1^2} \right) + \frac{e_1}{2m_1^2} \left[ i(\vec{q} \times \vec{p}) \cdot \langle \vec{S} \rangle_1 + (\vec{q} \cdot \vec{\epsilon}_1^*)(\vec{q} \cdot \vec{\epsilon}_1) \right], \quad (34)$$

while, by taking  $\mu = i$  in eq.(17), we get the spatial component as given by

$$J_{V(s=1)}^i = -e_1 \left[ \frac{\delta_1}{m_1} \vec{p}_i + \frac{i}{m_1} \epsilon_{ijk} \vec{q}_j \langle \vec{S}_k \rangle_1 \right], \quad (35)$$

which is very similar with its spin-1/2 counterpart, as we shall see in the following.

The spin-1 pseudo-vector current is simpler: according to its expression in eq.(16), we can obtain the following NR limits for  $\mu = 0$  and  $\mu = i$  as:

$$J_{PV(s=1)}^0 = \frac{ig_{PV}^1}{2m_1} \vec{q} \cdot \langle \vec{S} \rangle_1 \quad (36)$$

$$J_{PV(s=1)}^i = -\frac{g_{PV}^1}{2m_1^2} \left\{ \left[ \left( \vec{p} - \frac{1}{2} \vec{q} \right) \cdot \vec{\epsilon}_1 \right] (\vec{q} \times \vec{\epsilon}_1^*)_i - \left[ \left( \vec{p} + \frac{1}{2} \vec{q} \right) \cdot \vec{\epsilon}_1^* \right] (\vec{q} \times \vec{\epsilon}_1)_i \right\}. \quad (37)$$

We present below the corresponding fermionic currents with the purely vector cases also carrying the electromagnetic coupling constant:

$$J_{V(s=1/2)}^0 = e_1 \left\{ \delta_1 \left[ 1 + \frac{1}{4m_1^2} \left( \vec{p}^2 - \frac{1}{4} \vec{q}^2 \right) \right] + \frac{i}{4m_1^2} (\vec{q} \times \vec{p}) \cdot \langle \vec{\sigma} \rangle_1 \right\} \quad (38)$$

$$J_{V(s=1/2)}^i = e_1 \left[ \frac{\delta_1}{m_1} \vec{p}_i - \frac{i}{2m_1} \epsilon_{ijk} \vec{q}_j \langle \vec{\sigma}_k \rangle_1 \right], \quad (39)$$

$$J_{PV(s=1/2)}^0 = \frac{g_{PS}^1}{m_1} \vec{p} \cdot \langle \vec{\sigma} \rangle_1, \quad (40)$$

$$J_{PV(s=1/2)}^i = g_{PV}^1 \left\{ \left[ 1 - \frac{1}{4m_1^2} \left( \vec{p}^2 - \frac{1}{4} \vec{q}^2 \right) \right] \langle \vec{\sigma}_i \rangle_1 - \frac{i\delta_1}{4m_1^2} (\vec{q} \times \vec{p})_i + \frac{1}{2m_1^2} \left[ (\vec{p} \cdot \langle \vec{\sigma} \rangle_1) \vec{p}_i - \frac{1}{4} (\vec{q} \cdot \langle \vec{\sigma} \rangle_1) \vec{q}_i \right] \right\}. \quad (41)$$

Unlike the  $V$ - and  $PV$ - currents of the spin-1/2 case, the  $V$ - and  $PV$ - currents of the spin-1 field, as defined in Lagrangian (16) (marked by the  $V$  and  $PV$  coupling constants), always carry an explicit derivative. At high energies, this extra momentum factor must

dominate, but not in the NR limit. This extra derivative affects the way these currents interact with the gauge sector, specially in the case of spin-1. Furthermore, it is interesting to emphasize other differences between the fermionic and bosonic currents above. By comparing the vector currents, we notice essentially an extra term in the bosonic case associated with the contribution of the polarization,  $(\vec{q} \cdot \vec{\epsilon})(\vec{q} \cdot \vec{\epsilon}^*)$ . The  $PV$ -currents display more remarkable differences. For example, for the  $\mu = 0$  component, eqs.(36) and (40), we observe that the spin of the fermion couples to the average momentum,  $\vec{p}$ , while for the bosons the spin couples to the momentum transfer,  $\vec{q}$ . For the  $\mu = i$  components, we have many differences due to the spin terms (see eqs.(37) and (41)). These special features will be responsible for the different behaviors of the bosonic and fermionic potentials.

Let us now discuss the potentials involving  $V$ - and  $PV$ - currents mediated by a Proca-type particle. As already done in the previous Subsection, we shall exemplify the calculation through a particular configuration, and then quote the final results. We choose to calculate one of the simplest cases, the  $V - PV$  interaction between  $s = 1$  currents, since this case contains all the necessary elements to understand the other (lengthier) evaluations.

The amplitude is given by

$$\mathcal{A}_{V-PV}^{s=1} = i J_1^\mu \langle A_\mu A_\nu \rangle J_2^\nu = -\frac{1}{\vec{q}^2 + m_A^2} J_1^\mu J_{2\mu}, \quad (42)$$

where we have used current conservation,  $q^\mu J_\mu = 0$ , to eliminate the longitudinal contribution of the Proca-type propagator, eq.(4). In this particular case ( $s = 1$ ), both currents are conserved. We need therefore to simplify the contraction  $J_1^\mu J_{2\mu}$  and, according to our assumptions, only the  $J_1^0 J_{20}$  piece will contribute in the NR limit; so,

$$\mathcal{A}_{V-PV}^{s=1} \simeq -i \frac{e_1 g_{PV}^2 \delta_1}{2m_2} \frac{\vec{q} \cdot \langle \vec{S} \rangle_2}{\vec{q}^2 + m_A^2}, \quad (43)$$

and we finally calculate the Fourier integral to obtain

$$V_{V-PV}^{s=1} = -\frac{e_1 g_{PV}^2 \delta_1}{8\pi m_2} \frac{(1 + m_A r) e^{-m_A r}}{r^2} \hat{r} \cdot \langle \vec{S} \rangle_2. \quad (44)$$

In a similar way, the case with  $s = 1/2$  currents leads to

$$V_{V-PV}^{s=1/2} = -\frac{e_1 g_{PV}^2}{4\pi} \frac{e^{-m_A r}}{r} \left\{ \delta_1 \vec{p} \cdot \langle \vec{\sigma} \rangle_2 \left[ \frac{1}{m_1} + \frac{1}{m_2} \right] + \frac{(1 + m_A r)}{2m_1 r} [\langle \vec{\sigma} \rangle_1 \times \langle \vec{\sigma} \rangle_2] \cdot \hat{r} \right\} \quad (45)$$

and here the  $PV$ - current is not conserved; however, the conservation of the  $V$ - current ensures that  $q_\mu q_\nu$  term of the Proca propagator drops out.

As we have already seen in the discussion of the currents, the  $PV$ – currents of  $s = 1/2$  and  $s = 1$  exhibit remarkable differences and these explain why the potentials behave so differently. We anticipate that only this fermionic case yields the interaction with the  $\vec{p} \cdot \langle \vec{\sigma} \rangle$  dependence and the spin-spin contribution of the form  $\langle \vec{\sigma} \rangle_1 \times \langle \vec{\sigma} \rangle_2$ . In the bosonic case, we have the  $\hat{r} \cdot \langle \vec{S} \rangle$  dependence, but this is not exclusive of the  $V - PV$  interaction, it also appears in the  $S - PS$  bosonic potential. The other potentials,  $V - V$  and  $PV - PV$ , are listed below. First, the fermionic ones:

$$\begin{aligned}
V_{V-V}^{s=1/2} = & \frac{e_1 e_2}{4\pi} \frac{e^{-m_A r}}{r} \left\{ \delta_1 \delta_2 \left[ 1 + \frac{\vec{p}^2}{m_1 m_2} + \left( \frac{1}{4m_1^2} + \frac{1}{4m_2^2} \right) \left( \vec{p}^2 + \frac{m_A^2}{4} \right) \right] + \right. \\
& - \frac{(1 + m_A r)}{2r^2} \vec{L} \cdot \left[ \left( \frac{\delta_1 \langle \vec{\sigma} \rangle_2}{2m_2^2} + \frac{\delta_2 \langle \vec{\sigma} \rangle_1}{2m_1^2} \right) + \left( \frac{\delta_1 \langle \vec{\sigma} \rangle_2}{m_1 m_2} + \frac{\delta_2 \langle \vec{\sigma} \rangle_1}{m_1 m_2} \right) \right] + \\
& + \frac{1}{4m_1 m_2 r^2} \langle \vec{\sigma} \rangle_1 \cdot \langle \vec{\sigma} \rangle_2 (1 + m_A r + m_A^2 r^2) + \\
& \left. - \frac{1}{4m_1 m_2 r^2} (\langle \vec{\sigma} \rangle_1 \cdot \hat{r}) (\langle \vec{\sigma} \rangle_2 \cdot \hat{r}) (3 + 3m_A r + m_A^2 r^2) \right\}, \quad (46)
\end{aligned}$$

where we can read the electromagnetic potential by setting  $m_A = 0$ ; here  $e_1$  and  $e_2$  stand for the respective (electric) charges of the particles.

For the  $PV - PV$  case we obtained

$$\begin{aligned}
V_{PV-PV}^{s=1/2} = & -\frac{g_{PV}^1 g_{PV}^2}{4\pi} \frac{e^{-m_A r}}{r} \left\{ \langle \vec{\sigma} \rangle_1 \cdot \langle \vec{\sigma} \rangle_2 \left[ 1 - \frac{1}{4} \left( \vec{p}^2 + \frac{m_A^2}{4} + \frac{m_A}{2r} + \frac{1}{2r^2} \right) \left( \frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \right] + \right. \\
& + (\langle \vec{\sigma} \rangle_1 \cdot \vec{p}) (\langle \vec{\sigma} \rangle_2 \cdot \vec{p}) \left( \frac{1}{m_1 m_2} + \frac{1}{2m_1^2} + \frac{1}{2m_2^2} \right) + \\
& + \frac{(1 + m_A r)}{4r^2} \vec{L} \cdot \left( \frac{\delta_2 \langle \vec{\sigma} \rangle_1}{m_2^2} + \frac{\delta_1 \langle \vec{\sigma} \rangle_2}{m_1^2} \right) + \\
& \left. + \frac{1}{8r^2} (\langle \vec{\sigma} \rangle_1 \cdot \hat{r}) (\langle \vec{\sigma} \rangle_2 \cdot \hat{r}) \left( \frac{1}{m_1^2} + \frac{1}{m_2^2} \right) (3 + 3m_A r + m_A^2 r^2) \right\} + V_{PV-PV-LONG}^{s=1/2}, \quad (47)
\end{aligned}$$

where

$$\begin{aligned}
V_{PV-PV-LONG}^{s=1/2} \equiv & \frac{g_{PV}^1 g_{PV}^2}{4\pi} \frac{e^{-m_A r}}{r} \left\{ \left( \frac{3}{m_A^2 r^2} + \frac{3}{m_A r} + 1 \right) (\hat{r} \cdot \langle \vec{\sigma} \rangle_1) (\hat{r} \cdot \langle \vec{\sigma} \rangle_2) + \right. \\
& \left. - \left( \frac{1}{m_A^2 r^2} + \frac{1}{m_A r} \right) \langle \vec{\sigma} \rangle_1 \cdot \langle \vec{\sigma} \rangle_2 \right\}. \quad (48)
\end{aligned}$$

The fermionic  $PV - PV$  potential presents some peculiar aspects as an inheritance of the non-conserved  $PV$ – currents. First of all, in contrast to other cases, we had to take into account the longitudinal part of the Proca propagator in our calculations. By doing that, we obtain the contribution we have denoted by  $V_{PV-PV-LONG}^{s=1/2}$  in eq.(48). For this reason, this potential is not well defined in the massless limit ( $m_A \rightarrow 0$ ) of the exchanged particle



and the excitation of the unphysical spin-0 component of the massive vector boson ( $\partial_\mu A^\mu$  does not decouple any longer) jeopardises the unitarity of the model under study.

In order to give a consistent interpretation and circumvent the unitarity problem, we propose the following physical scenario: the exchanged boson is relatively heavy ( $m_1, m_2 > m_A \gg |\vec{q}|$ ) and we are bound to work at energy and momentum-transfer levels much below  $m_A$ , so that we are far under the threshold for the excitation of the unphysical longitudinal mode of  $A_\mu$  as justified by the hierarchy of masses stated above. We highlight two aspects concerning the potential specified by eqs. (47) and (48):

- We are working in the low momentum-transfer limit and, according to the uncertainty principle, one has the interaction range approximated by  $r \sim 1/|\vec{q}|$ . With this and the above hierarchy, we have  $r > \frac{1}{m_A} > \frac{1}{m_1}, \frac{1}{m_2}$ , which ensures that we are considering the potential for distances larger than the Compton wavelengths of the  $s = 1$  particles that act as external currents. This is consistent with the fact that the potential is a meaningful quantity for macroscopically large distances.
- Though our non-conserved pseudo-vector current couples to a massive vector boson, the fact that we are working in a low-energy regime (NR approximation, having in mind heavy  $m_1$  and  $m_2$ ) keeps us away from the danger of non-unitarity, which shows up in the high-energy domain ( $E \gg m_A$ ), so we conclude that the contribution described by eq.(48) is physically meaningful.

The spin-1  $V - V$  potential has the form:

$$\begin{aligned}
V_{V-V}^{s=1} = & \frac{e_1 e_2}{4\pi} \frac{e^{-m_A r}}{r} \left\{ \delta_1 \delta_2 \left[ 1 + \frac{\vec{p}^2}{m_1 m_2} + \left( \frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \left( \frac{\vec{p}^2}{2} - \frac{m_A^2}{8} \right) \right] + \right. \\
& + \frac{(1 + m_A r)}{r^2} \vec{L} \cdot \left[ \frac{1}{m_1 m_2} \left( \delta_1 \langle \vec{S} \rangle_2 + \delta_2 \langle \vec{S} \rangle_1 \right) + \left( \frac{\delta_1}{2m_2^2} \langle \vec{S} \rangle_2 + \frac{\delta_2}{2m_1^2} \langle \vec{S} \rangle_1 \right) \right] + \\
& + \frac{1}{m_1 m_2 r^2} \left[ \langle \vec{S} \rangle_1 \cdot \langle \vec{S} \rangle_2 (1 + m_A r + m_A^2 r^2) - (\hat{r} \cdot \langle \vec{S} \rangle_1) (\hat{r} \cdot \langle \vec{S} \rangle_2) (3 + 3m_A r + m_A^2 r^2) \right] + \\
& \left. - \zeta \left[ \frac{\delta_1}{2m_2^2 r^2} \left( \delta_2 (1 + m_A r) - (3 + 3m_A r + m_A^2 r^2) (\hat{r} \cdot \vec{\epsilon}_2) (\hat{r} \cdot \vec{\epsilon}_2^*) \right) + 1 \leftrightarrow 2 \right] \right\}, \quad (49)
\end{aligned}$$

where, as appears in eq.(30) for  $V_{S-S}^{s=1}$ , we have introduced the parameter  $\zeta$ . Again, its value in the vector field representation is  $\zeta = +1$ , whereas  $\zeta = -1$  in the case of the tensor field parametrization. We shall be more specific about this point in the next section. The

electromagnetic potential for the spin-1 charge carriers correspond to the choice  $m_A = 0$ , with  $e_1$  and  $e_2$  denoting the electric charges of the particles. In this case one clearly sees that the electromagnetic Coulomb interaction dominates at leading order for both spin-1/2 and spin-1 currents and, at this level, their respective potentials are indistinguishable.

Finally, the bosonic  $PV - PV$  potential

$$V_{PV-PV}^{s=1} = -\frac{g_{PV}^1 g_{PV}^2}{16\pi m_1 m_2} \frac{e^{-m_A r}}{r^3} \left\{ (3 + 3m_A r + m_A^2 r^2) (\hat{r} \cdot \langle \vec{S} \rangle_1) (\hat{r} \cdot \langle \vec{S} \rangle_2) + \right. \\ \left. - (1 + m_A r) \langle \vec{S} \rangle_1 \cdot \langle \vec{S} \rangle_2 \right\} \quad (50)$$

and here we recall that the  $PV -$  current for  $s = 1$  is conserved, contrary to what happens with  $s = 1/2$ . Therefore, this potential does not present the problems of its fermionic counterpart.

By comparing the results, we notice very similar spin-dependence in both  $V - V$  potentials, such as spin-orbit and spin-spin interactions. However, the bosonic case has additional terms with monopole-monopole and the polarization-polarization interactions (see the last term in eq.(49)). From the fermionic  $PV - PV$  potential, we observe an exclusive spin-spin interaction with a  $(\langle \vec{\sigma} \rangle_1 \cdot \vec{p})(\langle \vec{\sigma} \rangle_2 \cdot \vec{p})$ -dependence. The bosonic case does not present these interactions and it reveals the same functional spin-dependence of the  $PS - PS$  potential, eq.(32).

To conclude this Section, in possession of the potentials we have calculated for spin-1 charged currents/sources, we point out the possibility to apply our results to the study of the recently discovered heavy hyperhydrogen [31] and hyperhelium [32]. To study the details of the spectroscopy of the excited spin-1 states of  $\Lambda$ -hypernuclei, a careful analysis of the spin dependence of the potentials is needed (e.g. see Ref. [42]) and, in the particular case of the spin-1 states of the hypernuclei, such an analysis was actually lacking in the literature. Though we are not working out these applications in the present paper - it is out of our scope here - we endeavour to connect our results to the study of the excited spin-1/2 and spin-1 states of heavy hypernuclei. We shall report on that elsewhere.

#### IV. TENSOR FIELD REPRESENTATION

We have discussed above the spin-1 particle in terms of the familiar vector field representation, but this is not the only possibility [19][36]. On-shell, a massive charged spin-1 particle may well be described by a complex 2-form gauge field as given below:

$$\mathcal{L}_{tens.} = -\frac{1}{6}\mathcal{G}^{*\mu\nu\kappa}\mathcal{G}_{\mu\nu\kappa} + \frac{1}{2}m^2 B^{*\nu\kappa}B_{\nu\kappa} + ie(g-1)F_\nu{}^\mu B_{\mu\kappa}B^{*\nu\kappa}, \quad (51)$$

with  $\mathcal{G}_{\mu\nu\kappa} = D_\mu B_{\nu\kappa} + D_\kappa B_{\mu\nu} + D_\nu B_{\kappa\mu}$  being the associated gauge-covariant 3-form field-strength. The equations of motion are found to be

$$D^\mu \mathcal{G}_{\mu\nu\kappa} + m^2 B_{\nu\kappa} + ie(g-1)B_{\mu\kappa}F_\nu{}^\mu + ie(g-1)B_{\nu\mu}F_\kappa{}^\mu = 0. \quad (52)$$

Following similar arguments as in the Methodology, in the rest frame, we have

$$B_{rest}^{0i} = 0 \quad (53)$$

$$B_{rest}^{ij} = \epsilon_{ijk}\vec{\epsilon}_k, \quad (54)$$

where  $\vec{\epsilon}$  is the polarization 3-vector in the rest frame of the particle. By suitably applying a Lorentz boost we attain the final form for  $B^{\mu\nu}$  in the LAB system:

$$B^{0i} = \frac{N_B}{m}\epsilon_{ijk}\vec{\epsilon}_j\vec{P}_k \quad (55)$$

$$B^{ij} = N_B \left\{ \epsilon_{ijk}\vec{\epsilon}_k + \frac{1}{m(E+m)} \left[ \vec{P}_i(\vec{\epsilon} \times \vec{P})_j - \vec{P}_j(\vec{\epsilon} \times \vec{P})_i \right] \right\}, \quad (56)$$

where the second term in eq.(56) and the spatial component of eq.(13) can be shown to be analogous: it suffices to consider a specific direction and use the vector identity  $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$ . The normalization is given by  $N_B = 1/(2\sqrt{m})$ .

To study the matter sources and currents for this particular representation, we state below the interaction Lagrangian that contemplates all the couplings we are interested in:

$$\begin{aligned} \mathcal{L}_{int}^{s=1} = & \left[ g_S B_{\mu\nu}^* B^{\mu\nu} + g_{PS} B_{\mu\nu}^* \tilde{B}^{\mu\nu} \right] \phi + \\ & + ig_V [G^{*\mu\nu\kappa} B_{\nu\kappa} - G^{\mu\nu\kappa} B_{\nu\kappa}^* + (g-1)\partial^\nu (B_{\nu\alpha} B^{*\mu\alpha} - B_{\nu\alpha}^* B^{\mu\alpha})] A_\mu + \\ & + ig_{PV} [G^{*\mu\nu\kappa} \tilde{B}_{\nu\kappa} - G^{\mu\nu\kappa} \tilde{B}_{\nu\kappa}^*] A_\mu, \end{aligned} \quad (57)$$

where the coupling constants have the following canonical mass dimensions:  $[g_S] = [g_{PS}] = 1$  and  $[g_V] = [g_{PV}] = 0$ . The 3-form field-strength,  $G_{\mu\nu\kappa}$ , does not include the gauge-covariant

derivative. As previously discussed in connection with the gauge current in the vector field representation, the vector current we present below is the one to order  $e$  upon use of the subsidiary condition with the ordinary derivative, that is, the free subsidiary condition, which is compatible with our procedure to get the potential to order  $e^2$ .

Following this prescription, we obtained the same result as Delgado-Acosta *et al* [19] in terms of the  $\{p, q\}$  momentum variables, namely,

$$\mathcal{J}_V^\mu(p, q) = 2ep^\mu B_{\alpha\beta}^* B^{\alpha\beta} + iegq_\sigma B^{*\lambda\kappa} (\Sigma_T^{\mu\sigma})_{\lambda\kappa, \alpha\beta} B^{\alpha\beta}, \quad (58)$$

where

$$(\Sigma_T^{\kappa\sigma})_{\mu\nu, \alpha\beta} = \frac{1}{2} \left[ \eta_{\mu\alpha} (\Sigma_V^{\kappa\sigma})_{\nu\beta} - \eta_{\nu\alpha} (\Sigma_V^{\kappa\sigma})_{\mu\beta} + \eta_{\nu\beta} (\Sigma_V^{\kappa\sigma})_{\mu\alpha} - \eta_{\mu\beta} (\Sigma_V^{\kappa\sigma})_{\nu\alpha} \right] \quad (59)$$

is the spin generator of the tensor field representation and, from now on, the symbol  $\mathcal{J}$  shall denote the momentum-space sources/currents in the tensor field representation.

The NR sources/currents in this representation read:

$$\mathcal{J}_{S(s=1)} = \frac{g_S^1}{2m_1} \left\{ \delta_1 + \frac{1}{2m_1^2} \left[ i(\vec{q} \times \vec{p}) \cdot \langle \vec{S} \rangle_1 + \delta_1 \vec{q}^2 - (\vec{q} \cdot \vec{\epsilon}_1) (\vec{q} \cdot \vec{\epsilon}_1^*) \right] \right\} \quad (60)$$

$$\mathcal{J}_{PS(s=1)} = -\frac{ig_{PS}^1}{2m_1^2} \vec{q} \cdot \langle \vec{S} \rangle_1 \quad (61)$$

$$\mathcal{J}_{V(s=1)}^0 = e_1 \left\{ \delta_1 \left( 1 + \frac{\vec{p}^2}{2m_1^2} + \frac{\vec{q}^2}{8m_1^2} \right) - \frac{e_1}{2m_1^2} \left[ i(\vec{q} \times \vec{p}) \cdot \langle \vec{S} \rangle_1 + \delta_1 \vec{q}^2 - (\vec{q} \cdot \vec{\epsilon}_1) (\vec{q} \cdot \vec{\epsilon}_1^*) \right] \right\} \quad (62)$$

$$\mathcal{J}_{V(s=1)}^i = e_1 \left[ \frac{\delta_1}{m_1} \vec{p}_i + \frac{i}{m_1} \epsilon_{ijk} \vec{q}_j \langle \vec{S}_k \rangle_1 \right] \quad (63)$$

$$\mathcal{J}_{PV(s=1)}^0 = -\frac{ig_{PV}^1}{2m_1} \vec{q} \cdot \langle \vec{S} \rangle_1 \quad (64)$$

$$\mathcal{J}_{PV(s=1)}^i \sim \mathcal{O}(v^2). \quad (65)$$

The sources/currents of the two different spin-1 representations are very similar in the spin sector. Basically, a global sign distinguishes one from the other in every case, but this sign does not interfere in the evaluations of the amplitude. The only exception is the spatial component of the pseudo-vector current, where we notice some differences in the functional form. Nevertheless, this detail is irrelevant, because its contribution to the  $PV - PV$  or  $V - PV$  potentials leads to terms of order higher than two in the momenta, which we are currently ignoring in our calculations. For this reason, we refrain from giving the explicit (lengthy) form of  $\mathcal{J}_{PV(s=1)}^i$  and only indicate its order.

The remarkable differences appear in the polarization sector of the scalar source and time-component of the vector current. We recall that, in the vector field representation, eqs.(20) and (34), we have obtained the following polarization-dependence:

$$(\vec{q} \cdot \vec{\epsilon})(\vec{q} \cdot \vec{\epsilon}^*) = \vec{q}^2 \left( \frac{\vec{q}_i \vec{q}_j}{\vec{q}^2} \right) \vec{\epsilon}_i \vec{\epsilon}_j^*, \quad (66)$$

while in the tensor case, see eqs.(60) and (62), we have

$$\delta \vec{q}^2 - (\vec{q} \cdot \vec{\epsilon})(\vec{q} \cdot \vec{\epsilon}^*) = \vec{q}^2 \left( \delta_{ij} - \frac{\vec{q}_i \vec{q}_j}{\vec{q}^2} \right) \vec{\epsilon}_i \vec{\epsilon}_j^*. \quad (67)$$

According to these results, the two representations differ also due to opposite projections in the contribution coming from the polarization; this could be associated with the particular representation of the spin-1 we are dealing with. Let us be more specific: in order to evaluate the NR limit of these particular cases, we had to take into account the “weak” contributions (those with momenta) of the fields, such as the components  $W^0 \sim \vec{P} \cdot \vec{\epsilon}$ ,  $B^{0i} \sim (\vec{P} \times \vec{\epsilon})_i$ , see eqs.(13) and (56), respectively, and the analogues present in  $\vec{W}$  and  $B^{ij}$ . These different couplings between momenta and polarization,  $\vec{P} \cdot \vec{\epsilon}$  and  $\vec{P} \times \vec{\epsilon}$ , arise due to the Lorentz boost to the LAB frame and produce the observed longitudinal and transverse projections appearing in eqs.(66) and (67) above.

These results (eqs.(66) and (67)) raise a question concerning the vector or axial character of the spin-1 particle. In the  $W^\mu$ -case, we are assuming a vector-like particle. In the tensor representation, let us take the components as  $B^{0i} = -B_{0i} \equiv \vec{X}_i$  and  $B^{ij} = B_{ij} \equiv \epsilon_{ijk} \vec{Y}_k$ . On-shell, in the rest frame of the particle, all the degrees of freedom (d.f.) are carried by  $\vec{Y}$  (with  $\vec{X} = \vec{0}$ ). Going over to the LAB-system, the tensor representation gives  $\vec{X} \sim \vec{P} \times \vec{Y}$ . So, in a frame-independent way, the on-shell d.f. are actually described by  $\vec{Y}$ . The form of the vector current is insensitive to the vector or axial behavior of  $\vec{Y}$ , as we may conclude by a close inspection of the  $\mathcal{J}^\mu$  current expressed in terms of  $\vec{X}$  and  $\vec{Y}$ . If  $\vec{Y}$  is vector-like, then  $\vec{X}$  must be axial, and vice-versa. Therefore, we confirm that the minus sign difference highlighted above is always present as a consequence of the choice of representation (vector or tensor), regardless of whether the particle is vector- or axial-like.

At this point, it is instructive to discuss why we do not observe functional differences in all sources/currents, but only a global (numerical or mass) factor. For example, by comparing the spatial component of the vector currents, eqs.(35) and (63), we notice that the first contribution is of order  $\mathcal{O}(v)$ . One can check that the next correction should be

$\mathcal{O}(v^3)$ , which we are neglecting in this work. Similar arguments apply to the other cases. Therefore, we could only achieve possible different profiles in these currents if we consider the relativistic corrections to higher orders.

By using the same methodology as in Section III it can be shown that the  $V - PV$  and  $PV - PV$  potentials are identical to the results obtained in the vector field representation, eqs.(44) and (50), while the  $S - PS$  and  $PS - PS$  only differ by global mass factors,  $(1/2m_2^2)$  and  $(1/4m_1^2m_2^2)$ , respectively, due to the normalization and the different canonical mass dimension of the coupling constant  $g_{PS}$  (see eqs.(16) and (57)). The other two potentials,  $S - S$  and  $V - V$ , have, respectively, the following profiles:

$$V_{S-S}^T = V_{S-S}^{s=1} \Big|_{\zeta=-1} + \frac{g_S^1 g_S^2}{16\pi m_1 m_2} \frac{e^{-m_\phi r}}{r} \left[ \delta_1 \delta_2 \frac{m_\phi^2}{2} \left( \frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \right] \quad (68)$$

$$V_{V-V}^T = V_{V-V}^{s=1} \Big|_{\zeta=-1} + \frac{e_1 e_2}{4\pi} \frac{e^{-m_A r}}{r} \left[ \delta_1 \delta_2 \frac{m_A^2}{2} \left( \frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \right]. \quad (69)$$

The difference between the two representations in both  $S - S$  and  $V - V$  cases appears as monopole-monopole terms and results in opposite behaviors in the polarization-polarization sector, since  $\zeta = +1$  and  $-1$  in the vector and tensor representation, respectively. Actually, these are consequences of what we have seen in the discussion above about the scalar **source** and time-component of the vector current.

In general, we cannot distinguish between the vector and tensor field representations for the  $s = 1$  by only considering the spin-dependent sector of the potentials. We emphasize that the differences are suppressed because they only show up in order  $\mathcal{O}(v^2)$  in the amplitude. This is so because we are here bound to consider only dipole contributions. If we extend our calculations to also include quadrupole effects, the differences in the  $V - V$  potentials for the vector and tensor representation of the spin-1 might become evident, as we could expect from the studies reported in Ref. [19]. The consequences of the quadrupole moments of the two representations are interesting to be investigated and we intend to report on that in a forthcoming work [43].

## V. CONCLUDING REMARKS

In this paper, we focused our attention on the special properties of the spin-spin dependence of the source/current–source/current interaction potential in the non-relativistic limit

for spin-1 and spin-1/2 charged matter particles. Our results indicate some universalities between bosons and fermions when exchanging scalar and vector particles. Essentially, we observe very similar contributions in the spin-sector of the  $S - S$ ,  $S - PS$ ,  $PS - PS$  and  $V - V$  potentials. The main differences show up in connection with  $PV$ - currents, namely, the  $V - PV$  and  $PV - PV$  potentials.

We highlight here a result presented in Section IV: the non-trivial consequences of choosing a particular field representation for the spin-1 particle. Even if the vector and tensor representations are equivalent on-shell (and the particles in the source/current are on-shell), the results for the  $S - S$  and  $V - V$  potentials do differ. The justification was given in the aforementioned Section and we claim that, by including higher-order terms in the relativistic corrections, quadrupole moments appear and the differences become explicit.

Furthermore, we do not present a discussion on the bosonic and fermionic (pseudo-)tensor currents exchanging a 2-form gauge field. However, according to preliminary results of a work in progress, it is worthwhile to comment that the case of bosonic sources/currents would yield interesting new interactions involving spin-polarization couplings, e.g.  $\vec{\epsilon}_1 \cdot \langle \vec{S} \rangle_2$ .

To close this report, we point out a question that might be addressed to with the help of our results on the comparison between the classes of potentials describing the interaction between spin-1/2 and spin-1 matter sources/currents. For the fermionic case, the electromagnetic potential does not couple to the particle density, as it happens instead for spin-0 and spin-1 charged matter.

If a charged spin-1 field,  $W^\mu$ , is non-minimally coupled (see eq.(10)) to the electromagnetic field, we have

$$\begin{aligned} \partial_\mu F^{\mu\nu} - 2e^2 (W_\mu^* W^\mu) A^\nu + e^2 [W^{*\nu} (W_\mu A^\mu) + W^\nu (W_\mu^* A^\mu)] = \\ = J_{global}^\nu + ie\partial_\mu (W^\mu W^{*\nu} - W^{*\mu} W^\nu) \end{aligned} \quad (70)$$

where  $J_{global}^\nu = -ie (W_\mu^* W^{\nu\mu} - W^{*\nu\mu} W_\mu)$  and  $W_{\nu\mu} \equiv \partial_\nu W_\mu - \partial_\mu W_\nu$ . The second term of the RHS contributing to  $J^\nu$  stems from the non-minimal coupling from eq.(10) and is mandatory to ensure  $g = 2$  for the charged spin-1 boson.

From eq.(70), two interesting properties of the electromagnetic interactions of spin-1 matter fields are made explicit: the London-type term that couples  $A^\mu$  to the density of carriers and the new interaction between the photon and the charged boson through their polarization vectors. These special aspects are expected to influence the inter-particle potential only

if we take loop effects into account.

However, the  $e^2$ -terms in eq.(70) introduce singularities associated to the point-like idealization of spin-1 charged matter particles. The 1962 paper by Dirac [44] and the recent works by Fabbri [45] and Dain [46] address the issue of extensibility in connection with charged particles.

In the present paper, we do not calculate classical field configurations generated by spin-1 currents; instead, we build up interaction potentials by means of a semi-classical calculation, so we do not run into the complications yielded by the singularities mentioned above. It would be nevertheless worthwhile to analyse the details of the connection between particle extension, mass, charge and spin. The electrodynamics of bosonic carriers seems to suggest that the point-like idealization of charged particles is indeed very restrictive.

We hope to carry on with our studies in this direction and to make use of our results for the spin-1 potentials to get a deeper insight in the understanding of the electrodynamical properties of charged bosonic fields in the classical regime.

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